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JENSEN MEASURES OF UNIFORM ALGEBRAS

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ABSTRACT
It is proved that every Jensen measure of any uniform algebra is the distinguished member of a certain system of Jensen measures. The system has several fundamental properties common to measures that are induced from Brownian motions by means of the stopping time.

1. Introduction

Let A be any uniform algebra on some compact space. The maximal ideal space of A is denoted by Y. We use the letter J to denote the convex cone of A-subharmonic functions in $C_R(Y)$, which are stable in the max-operation (cf. [11]). The cone J defines the partial order relation $\prec$ among finite positive regular Borel measures $M^+(Y)$ on Y.

Jensen measures were defined by E. Bishop [3] in the framework of general uniform algebra theory. He also showed the existence of nontrivial Jensen measures. A probability measure $\mu \in M^+(Y)$ is said to be a Jensen measure for $p \in Y$ if it satisfies

$$\log|f(p)| \leqq \int \log|f(y)|\mu(dy), \qquad \forall f \in A.$$

It is easy to check that a positive measure μ is a Jensen measure for p, if and only if it satisfies $\delta_p \prec \mu$, where δ_p is the point mass at p.

We introduced in [13] the notion of consistency in the context of a uniform algebra to define a certain subclass of Jensen measures. Each measure of that

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class was called terminal measure, which was the special member in the following system of measures. For each point p of Y, the symbol F_p denotes the totality of compact sets in Y that contain p. A subset $(\lambda_G : G \in F_p)$ of $M^+(Y)$ indexed by F_p is called a consistent system (with base point p), if it satisfies

(a) each λ_G is a Jensen measure for p carried on G;

(b) if $F \subset G$, then $\lambda_F \prec \lambda_G$, i.e. $\lambda_F(k) \leqq \lambda_G(k) = \int k(y)\lambda_G(dy)$ for all $k \in J$;

(c) if $F \subset G$, we have $\lambda_F(g) \leqq \lambda_G(g)$ for any nonnegative function $g \in C_R(Y)$ such that $g = 0$ on $G \setminus F$.

(The above definition is slightly different in form from the original one. But the present style is rather desirable for the purpose here.)

An arbitrary consistent system $(\lambda_G : G \in F_p)$ obviously contains the distinguished measure λ_Y which is maximum in the system with respect to the order $\prec$. By terminal measure we mean this unique measure λ_Y. For each $p \in Y$ we denote by T_p the totality of terminal measures of consistent systems with base point p.

The notion of consistency was required to generalize probabilistic theory of Hardy spaces in the framework of a uniform algebra. Any terminal measure admits a certain maximal function to each nonnegative function in J. This maximal function is the analogue of the Brownian maximal function well-known in classical Hardy spaces. Our maximal function satisfies, together with its original function, Doob's maximal inequality [8], Burkholder–Gundy–Silverstein inequalities [4] and Fefferman–Stein inequality [9] as well as other familiar estimates. Furthermore terminal measures are fairly easy to deal with, though their definition seems complicated: for any A and any $p \in Y$ there always exists a consistent system with base point p whose terminal measure is carried on the Shilov boundary of A. Also the notion of consistency applied to general theory of compact convex sets leads us to the conclusion that any probability measure on a compact convex set is the terminal measure of some consistent system, where we use the cone of continuous convex functions on the set to define the partial order such as $\prec$ (cf. [14]). There is another important information by B. Cole concerning Riesz estimates, etc. (cf. [11]). From these results one may guess every Jensen measure of any A to be the terminal measure of a consistent system. Fortunately our conjecture is true, which is the purpose in this paper. Here it should be noticed that the similar conclusion cannot be expected from the totality of representing measures because of H. König [12], also cf. K. Yabuta [15].

Finally the author should like to express his deep gratitude to Professor T. W. Gamelin who gave him very affirmative suggestions on the above matter.

2. Jensen measures of polydisk algebras

The aim in this section is to prove that every Jensen measure of polydisk algebras $A(U^n)$ is the terminal measure of a consistent system. We start by reconfirming stability properties of terminal measures. The following proposition is essentially the same as in [13].

PROPOSITION 2.1. *For any uniform algebra A and any point $p \in Y$ the set T_p is weakstar compact convex in the dual of $C_R(Y)$.*

Let $(\lambda_Y^i)_{i\in E}$ be a set of terminal measures associated with A, whose index set E is directed upward. Suppose λ_Y^i converges vaguely along E. Then its limit is the terminal measure of a consistent system.

SKETCH OF THE PROOF. The first assertion is immediate from very definition of consistency.

Let $(\lambda_G^i : G \in F_{p_i})$ be a consistent system whose terminal measure is equal to λ_Y^i in question. Since λ_Y^i is convergent, $(p_i)_{i\in E}$ also converges to some point $p \in Y$. Applying an ultra filter I on E finer than the section filter, we put

$$\lambda_G = \begin{cases} \lim\limits_I \lambda_G^i & \text{if } p \in G^0 = \operatorname{Int} G, \\ \delta_p & \text{otherwise}, \quad G \in F_p. \end{cases}$$

It is easily seen that $\lambda_Y = \lim_E \lambda_Y^i$ and $(\lambda_G : G \in F_p)$ is consistent.

The above observation yields that for each $g \in C_R(Y)$ the function

$$\tilde{g}(y) = \inf\{\mu(g) : \mu \in T_y\}, \qquad \forall y \in Y$$

is lower semi-continuous on Y. Clearly $\| \tilde{g} \|_\infty \leqq \| g \|_\infty$. On the other hand, we have $\tilde{g} \leqq g$ on Y because $(\lambda_G : G \in F_p)$, $\forall \lambda_G = \delta_p$, is consistent.

Our next task is to prove $\mu(\tilde{g}) \geqq \tilde{g}(y)$ for any $\mu \in T_y$ and $g \in C_R(Y)$. We first show a rough sketch of our strategy. For a fixed g there is a set $\{(\lambda_{y,G} : G \in F_y)\}_{y\in Y}$ of consistent systems with $\lambda_{y,Y}(g) = \tilde{g}(y)$. If $(\lambda_{y,G})_{y\in G}$ were always a sweeping system in the sense of Constantinescu–Cornea [7], or equivalently if the function $\lambda_{y,G}(h)$ in the variable y were Borel measurable on G for every $h \in C_R(Y)$, we could sweep each measure $\nu \in M^+(Y)$ to get $S_G\nu \in M^+(Y)$ satisfying

$$S_G\nu(h) = \int_{Y\setminus G^0} h(y)\nu(dy) + \int_{G^0} \lambda_{y,G}(h)\nu(dy).$$

Then the sweeping $(S_G\lambda_G : G \in F_y)$ of a consistent system $(\lambda_G : G \in F_y)$ would be consistent and satisfy $\lambda_Y(\tilde{g}) = S_Y\lambda_Y(g) \geqq \tilde{g}(y)$. But in the present situation such a property cannot be expected from $(\lambda_{y,G})_{y\in G}$. To avoid this difficulty, we apply the ultra filter technique.

Let D denote a decomposition of Y into a finite number of pairwise disjoint Borel sets. The totality $\mathcal{D}$ of such D forms an upper directed set in the following order $\leqq$:

$$D \leqq \tilde{D} \text{ if any } \tilde{E} \in \tilde{D} \text{ is included in some } E \in D.$$

We pick up and fix an ultra filter I on $\mathcal{D}$ which is finer than the section filter associated with $\leqq$. Moreover, for each nonempty element E of $D \in \mathcal{D}$ we fix a point p_E of E. Let G be a nonempty compact subset of Y and let $\nu \in M^+(Y)$. For any $D \in \mathcal{D}$ with $(Y \setminus G^0, G^0) \leqq D$ we put, using a given set of consistent systems $\{(\lambda_{y,G} : G \in F_y)\}_{y\in Y}$,

$$S_{G,D}\nu = \nu \mid (Y \setminus G^0) + \sum_{\phi \neq E \in D,\, E \subset G^0} \nu(E) \cdot \lambda_{p_E,G}.$$

It is easy to check that $S_{G,D}$ is a norm-preserving affine map from $M^+(Y)$ into itself, and $S_{G,D}\nu$ is carried on G if and only if ν is a measure on G. Also $S_{G,D}\nu = \nu$ if $\nu(G^0) = 0$. These are still valid for the map S_G:

$$M^+(Y) \ni \nu \to S_G\nu := \text{w*-}\lim_I S_{G,D}\nu.$$

LEMMA 2.2. *The family* $(S_G : G \in \bigcup_{y\in Y} F_y)$ *of affine maps of* $M^+(Y)$ *into itself possesses the following properties*:

(1) *If* $\lambda_{y,Y}(g)$ *is* ν-*integrable, we have* $S_Y\nu(g) = \int \lambda_{y,Y}(g)\nu(dy)$ *where* $g \in C_R(Y)$ *and* $\nu \in M^+(Y)$;

(2) $\nu \prec S_G\nu$ *for every* $\nu \in M^+(Y)$ *and any nonempty compact set* $G \subset Y$;

(3) *if* $\nu \in M^+(Y)$ *has no mass on* $Y \setminus F^0$ *and if* $F \subset G$, *we have* $S_F\nu \prec S_G\nu$ *and* $S_F\nu(g) \leqq S_G\nu(g)$ *for each nonnegative* $g \in C_R(Y)$ *such that* $g \mid G \setminus F = 0$;

(4) *the sweeping* $(S_G\lambda_G : G \in F_p)$ *of any consistent system* $(\lambda_G : G \in F_p)$ *is consistent*.

PROOF. (1) For any strictly positive number ε there exists a compact set $K \subset Y$ such that $\lambda_{y,Y}(g) \mid K$ is continuous and $\nu(Y \setminus K) \| g \|_\infty \leqq \varepsilon$ by Egoroff's lemma. If $D \in \mathcal{D}$ satisfies $(K, Y \setminus K) \leqq D$, we get

$$S_{Y,D}\nu(g) = \sum_{\phi \neq E \in D} \nu(E)\lambda_{p_E,Y}(g)$$

$$\geqq \sum_{\phi \neq E \in D,\, E \subset K} \nu(E)\lambda_{p_E,Y}(g) - \varepsilon$$

$$\geqq \int \lambda_{y,Y}(g)\nu(dy) - 2\varepsilon - \nu(K)$$

$$\times \left(\sup_{\substack{\phi \neq E \in D \\ E \subset K}} \left\{ \sup_{z,y \in E} \{ |\lambda_{y,Y}(g) - \lambda_{z,Y}(g)| \} \right\} \right).$$

Since the last term becomes smaller as D gets large, the above yields

$$S_Y\nu(g) \geqq \int \lambda_{y,Y}(g)\nu(dy) - 2\varepsilon, \qquad \text{so that } S_Y\nu(g) \geqq \int \lambda_{y,Y}(g)\nu(dy),$$

because ε is arbitrary. Repeating the same argument with $-g$ in place of g, we obtain the converse inequality, and hence

$$S_Y\nu(g) = \int \lambda_{y,Y}(g)\nu(dy).$$

(2) We first notice that $k(y) \leqq \lambda_{y,G}(k)$ whenever $k \in J$ and $y \in G$. Therefore

$$S_{G,D}\nu(k) = \int_{Y \setminus G^0} k(y)\nu(dy) + \sum_{\phi \neq E \in D,\, E \subset G^0} \nu(E)\lambda_{p_E,G}(k)$$

$$\geqq \int_{Y \setminus G^0} k(y)\nu(dy) + \sum_{\phi \neq E \in D,\, E \subset G^0} k(p_E)\nu(E)$$

$$\geqq \int k(y)\nu(dy) - \nu(G^0)\left(\sup_{\substack{\phi \neq E \in D \\ E \subset G^0}} \left\{ \sup_{z,y \in E} \{ |k(z) - k(y)| \} \right\} \right),$$

taking the limit to get $\nu(k) \leqq S_G\nu(k)$, i.e. $\nu \prec S_G\nu$.

(3) Under the present situation any $k \in J$ allows

$$S_{F,D}\nu(k) = \sum_{\phi \neq E \in D,\, E \subset F^0} \nu(E)\lambda_{p_E,F}(k) \leqq \sum_{\phi \neq E \in D,\, E \subset F^0} \nu(E)\lambda_{p_E,G}(k)$$

$$= \sum_{\phi \neq E \in D,\, E \subset G^0} \nu(E)\lambda_{p_E,G}(k) = S_{G,D}\nu(k)$$

whenever D is large enough. This is due to the fact that $\nu(E) = 0$ with E outside F^0 and $\lambda_{p_E,F} \prec \lambda_{p_E,G}$. Accordingly it turns out that $S_F\nu \prec S_G\nu$. Furthermore if

$g \in C_R(Y)$ is nonnegative and vanishes on $G \setminus F$, we have $\lambda_{y,F}(g) \leqq \lambda_{y,G}(g)$, $\forall y \in F$. Hence

$$S_{F,D}\nu(g) = \sum_{\phi \neq E \in D,\, E \subset F^0} \nu(E)\lambda_{p_E,F}(g)$$

$$\leqq \sum_{\phi \neq E \in D,\, E \subset F^0} \nu(E)\lambda_{p_E,G}(g) \leqq S_{G,D}\nu(g)$$

and so $S_F\nu(g) \leqq S_G\nu(g)$.

(4) Let $(\lambda_G : G \in F_p)$ be any consistent system. It follows from (2) that $\delta_p \prec \lambda_G \prec S_G\lambda_G$, so that $S_G\lambda_G$ is a Jensen measure for p carried on G. Next suppose $G \in F_p$ includes $F \in F_p$. Here recall that condition (c) in the definition of consistency is equivalent to saying $\lambda_F \mid V \leqq \lambda_G$, where V is the relative interior of F with respect to G. This implies

$$S_F[\lambda_F \mid \partial F] = \lambda_F \mid \partial F \prec \lambda_G - \lambda_F \mid F^0 \prec S_G[\lambda_G - \lambda_F \mid F^0]$$

so that by the former half of (3)

$$S_F\lambda_F = \lambda_F \mid \partial F + S_F[\lambda_F \mid F^0] \prec S_G[\lambda_G - \lambda_F \mid F^0] + S_G[\lambda_F \mid F^0] = S_G\lambda_G.$$

Moreover if $g \in C_R(Y)$ is nonnegative and vanishes on $G \setminus F$, we get

$$\int_{\partial F} g(y)\lambda_F(dy) \leqq \int_{\partial G} g(y)\lambda_G(dy), \qquad \text{since } \lambda_F \mid \partial F \cap V \leqq \lambda_G \mid \partial G.$$

From the similar reason and by (3) it follows that

$$S_F[\lambda_F \mid F^0](g) \leqq S_G[\lambda_F \mid F^0](g) \leqq S_G[\lambda_F \mid F^0](g) + S_G[\lambda_G \mid G^0 - \lambda_F \mid F^0](g)$$

$$= S_G[\lambda_G \mid G^0](g).$$

Combining two inequalities, we conclude $S_F\lambda_F(g) \leqq S_G\lambda_G(g)$. Thus $(S_G\lambda_G : G \in F_p)$ is surely consistent.

Proposition 2.3. *Let g be any function in $C_R(Y)$. For any $p \in Y$ and each $\mu \in T_p$ we have $\mu(\tilde{g}) \geqq \tilde{g}(p)$.*

Proof. With the aid of Proposition 2.1 there exists a family $\{(\lambda_{y,G} : G \in F_y)\}_{y \in Y}$ of consistent systems such that $\lambda_{y,Y}(g) = \tilde{g}(y)$. Let $(S_G : G \in \bigcup_{y \in Y} F_y)$ be the family of affine maps of $M^+(Y)$ into itself which is induced from $\{(\lambda_{y,G} : G \in F_y)\}_{y \in Y}$ and the ultra filter I on $\mathscr{D}$. Then for any consistent system $(\lambda_G : G \in F_p)$ with base point p, the system $(S_G\lambda_G : G \in F_p)$ is consistent by Lemma 2.2. In particular $S_Y\lambda_Y(g) \geqq \tilde{g}(p)$ by very definition of $\tilde{g}$. On the other hand, it follows from (1) of Lemma 2.2 that

$$S_Y\lambda_Y(g) = \int \lambda_{y,Y}(g)\lambda_Y(dy) = \int \tilde{g}(y)\lambda_Y(dy)$$

since $\tilde{g}$ is lower semi-continuous. Hence we get $\lambda_Y(\tilde{g}) \geqq \tilde{g}(p)$.

PROPOSITION 2.4. *Let K be a nonempty A-convex compact set in Y and let A_K be the closure in $C(Y)$ of the restriction algebra $A \mid K$ of A onto K. The terminal measure of a consistent system associated with A_K is the terminal measure of some consistent system with respect to A.*

PROOF. We first notice that the maximal ideal space of A_K can be identified with K. Let $(\mu_E)_{p \in E \subset K, E \text{ compact}}$ be any consistent system of A_K with base point $p \in K$. We show that μ_K is the terminal measure of some consistent system $(\lambda_G : G \in F_p)$ with respect to A. For each $G \in F_p$ call $\lambda_G = \mu_{G \cap K}$. Each λ_G is a Jensen measure for p, which is carried on G and associated with A. If $G \in F_p$ includes $F \in F_p$, we get by $F \cap K \subset G \cap K$

$$\lambda_F = \mu_{F \cap K} \prec \mu_{G \cap K} = \lambda_G,$$

since $J \mid K$ is dense in continuous A_K-subharmonic functions on K. Furthermore, if a nonnegative function $g \in C_R(Y)$ vanishes on $G \setminus F$, it also vanishes on $(G \cap K) \setminus (F \cap K)$, and hence we get

$$\lambda_F(g) = \mu_{F \cap K}(g) \leqq \mu_{G \cap K}(g) = \lambda_G(g).$$

Thus $(\lambda_G : G \in F_p)$ is a consistent system whose terminal measure is equal to $\mu_{Y \cap K} = \mu_K$.

Let U^n be the unit open polydisk in $\mathbf{C}^n$. For a positive number r, rU^n is the subset $(rz : z \in U^n)$ of $\mathbf{C}^n$. The polydisk algebra $A(U^n)$ is defined as the closure in $C(\overline{U^n})$ of analytic polynomials on $\mathbf{C}^n$. It is known, and easy to check that the maximal ideal space of $A(U^n)$ is identical with $\overline{U^n}$ (cf. [10]).

LEMMA 2.5. *Let $C \subset \overline{U^n}$ be a circle which lies on some analytic line in $\mathbf{C}^n$. The normalized Lebesgue measure on C is the terminal measure of a consistent system of $A(U^n)$, whose base point is the center of C.*

PROOF. The linearly convex hull K of C is obviously a closed disk in $\overline{U^n}$, which is included in the analytic line. It is easily seen that K is $A(U^n)$-convex and $A(U^n)_K$ can be identified with the disk algebra $A(K)$. On the other hand, the normalized Lebesgue measure on C is the terminal measure of a consistent system of $A(K)$ whose base point is the center of C ([13], cf. [14]). Hence by Proposition 2.4 we obtain the assertion.

PROPOSITION 2.6. *Every Jensen measure of $A(U^n)$ is the terminal measure of a consistent system.*

PROOF. We first suppose that a given Jensen measure μ for $p \in \overline{U^n}$ is carried on $\overline{(1-r)U^n}$, where r is a number in the interval $(0, 1)$. Note that $\overline{(1-r)U^n}$ is $A(U^n)$-convex. In particular $p \in \overline{(1-r)U^n}$. Assume that μ is not contained in T_p. Since T_p is weakstar compact convex by Proposition 2.1, there is a function $g \in C_R(\overline{U^n})$ such that

$$\tilde{g}(p) = \inf\{\nu(g) : \nu \in T_p\} > \mu(g),$$

by separation theorem. Let ψ be a nonnegative smooth function on $\mathbf{C}^n$ which is carried on U^n and satisfies $\int \psi(q)V(dq) = 1$, where V is $2n$-dimensional Lebesgue measure defined on $\mathbf{C}^n$. For each V-integrable function h and any positive number ε we put

$$h_\varepsilon(z) = \int h(z-q)\varepsilon^{-2n}\psi(\varepsilon^{-1}q)V(dq).$$

Here we may suppose g and $\tilde{g}$ are respectively defined on $\mathbf{C}^n$ to be identically zero outside $\overline{U^n}$. We have then $\tilde{g}_\varepsilon \leqq g_\varepsilon$ on $\mathbf{C}^n$ and $g_\varepsilon \to g$ locally uniformly on U^n as $\varepsilon \to 0$. We further require that $\tilde{g}_\varepsilon$ is plurisubharmonic on a neighbourhood of $\overline{(1-r)U^n}$ for each ε in $(0, r/2)$. Let $C \subset \overline{(1-r/2)U^n}$ be a circle with center z, which lies on an analytic line $L \subset \mathbf{C}^n$, and γ be the normalized Lebesgue measure on C. For any w of $\overline{(r/2)U^n}$ it is easily seen that $E \to \gamma(E+w)$ on Baire sigma algebra over $C - w$ gives the normalized Lebesgue measure with respect to the circle $C - w$. Hence by Lemma 2.5 and from Proposition 2.3 we get

$$\begin{aligned}\int \tilde{g}_\varepsilon(q)\gamma(dq) &= \int \varepsilon^{-2n}\psi(\varepsilon^{-1}w)V(dw)\int \tilde{g}(q-w)\gamma(dq)\\ &\geqq \int \tilde{g}(z-w)\varepsilon^{-2n}\psi(\varepsilon^{-1}w)V(dw)\\ &= \tilde{g}_\varepsilon(z).\end{aligned}$$

Since C is arbitrary, the above implies that $\tilde{g}_\varepsilon$ is plurisubharmonic on

$(1 - r/2)U^n$. By Bremermann's theorem [4] $\tilde{g}_\varepsilon$ is uniformly approximated on $\overline{(1 - r)U^n}$ by functions in J (cf. [11]). This yields

$$\mu(g_\varepsilon) \geqq \mu(\tilde{g}_\varepsilon) \geqq \tilde{g}_\varepsilon(p).$$

Lettting $\varepsilon \to 0$, we get

$$\mu(g) \geqq \liminf_{\varepsilon \to 0} \tilde{g}_\varepsilon(p) \geqq \tilde{g}(p).$$

The last inequality is due to the fact that $\tilde{g}$ is lower semi-continuous at $p \in U^n$. Thus we reached a contradiction:

$$\mu(g) \geqq \tilde{g}(p) > \mu(g).$$

So μ must belong to T_p.

Finally an arbitrary Jensen measure μ for a point $p \in \overline{U^n}$ is vaguely approximated by the measures μ_r on $\overline{rU^n}$ defined as

$$\mu_r(g) = \int g(rq)\mu(dq), \qquad \forall g \in C_R(\overline{U^n}), \quad 0 < r < 1.$$

Since μ_r is a Jensen measure for rp, we conclude by the result above and Proposition 2.1 that μ is the terminal measure of some consistent system associated with $A(U^n)$.

The following proposition will be necessary in the next section.

PROPOSITION 2.7. *Let A be any uniform algebra with the maximal ideal space Y. If two measures $\mu, \nu \in M^+(Y)$ satisfy $\nu \prec \mu$, ν is carried on the A-convex hull in Y of the support of μ. In particular measures in a consistent system are carried on the A-convex hull of the support of their terminal measure.*

PROOF. Let K be the A-convex hull of the support of μ. Assume that $\nu(Y \setminus K) > 0$. There exists a compact set $F \subset Y$ disjoint from K, where ν has a positive mass. Since K is A-convex, there is a function $f \in A$ for any point $y \in F$ such that

$$\sup_{z \in K} \{|f(z)|\} \leqq 1 < f(y).$$

This implies that J contains a function k satisfying

$$\sup_{z \in K} \{k(z)\} \leqq 0 < \inf_{y \in F} \{k(y)\}.$$

On the other hand, by Cartier–Fell–Meyer's theorem, there exists a decomposition $\mu = \mu_1 + \mu_2$ of μ into positive measures such that $\nu \mid F \prec \mu_1$ and $\nu \mid (Y \setminus F) \prec \mu_2$ ([6], cf. [1]). But this yields

$$0 < [\nu \mid F](k) \leqq \mu_1(k) \leqq 0,$$

a contradiction. Hence ν is carried on K.

3. Uniform algebras

We shall prove in this final section that every Jensen measure of any A is the terminal measure of some consistent system on Y. The argument will go as follows. Any closed subalgebra of A generated by a finite set in A can be identified with the algebra on the compact subset K of $\mathbf{C}^n$, which is the image of Y under the canonical map. Each Jensen measure μ of A defines a unique Jensen measure ν on K. From the result already proven and by Arens–Calderón lemma, we shall manufacture a certain consistent system, *carried on* K, whose terminal measure is equal to ν.

We first show how the fact above is used to derive our main theorem. For any subset Q of A we denote by $C_R[Q]$ the closed subalgebra of $C_R(Y)$ generated by constants and $\{\mathrm{Re}\, f, \mathrm{Im}\, f : f \in Q\}$. $C_R[Q]$ is a Banach lattice with order unit 1.

Lemma 3.1. *Let μ be a Jensen measure of A which represents the evaluation functional at $p \in Y$. Suppose any nonempty finite subset Q of A allows a system $(L_{G,Q} : G \in F_p)$ of positive linear forms on $C_R[Q]$ with norm 1 such that*

(1) *each $L_{G,Q}$ is carried on G, i.e. $L_{G,Q}(g) = 0$ if $g \in C_R[Q]$ vanishes on G;*
(2) *if $F \subset G$, we have $k(p) \leqq L_{F,Q}(k) \leqq L_{G,Q}(k)$ for each $k \in C_R[Q]$ of the form:*

$$\sup\{c_j \log |f_j| : R \ni c_j \geqq 0, f_j \in Q\};$$

(3) *inclusion $F \subset G$ implies $L_{F,Q}(g) \leqq L_{G,Q}(g)$ for any nonnegative $g \in C_R[Q]$ with $g \mid G \setminus F = 0$;*
(4) *$\mu \mid C_R[Q] = L_{Y,Q}$.*

Then μ is the terminal measure of a consistent system on Y.

Proof. The totality $\mathscr{D}$ of nonempty finite sets $Q \subset A$ forms an upper directed set under the set theoretic inclusion $\subset$. Let I be an ultra filter on $\mathscr{D}$ finer than the section filter of $\subset$. Call

$$L_G^*(g) = \lim_I L_{G,Q}(g), \qquad g \in \bigcup_{Q \in \mathscr{D}} C_R[Q], \quad G \in F_p.$$

Obviously L_G^* uniquely extends to the positive linear form λ_G on $C_R(Y)$ with norm 1. Here recall that each function of J is uniformly approximated on Y by functions in the form (cf. [11]):

$$\sup\{c_j \log |f_j| : R \ni c_j \geqq 0, f_j \in A\}.$$

In particular functions of the form as in (2) are dense in J. This implies together with (2) that each λ_G is a Jensen measure for p and that $\lambda_F \prec \lambda_G$ if $F \subset G$.

Next, suppose $g \in C_R(Y)$ is nonnegative and vanishes on $G \setminus F$, where $F, G \in F_p$ and $F \subset G$. Since $\bigcup_{Q \in \mathscr{D}} C_R[Q]$ is dense in $C_R(Y)$, there exists a finite set $Q_0 \subset A$ and a function $h \in C_R[Q_0]$ such that $\| h - g \|_\infty \leqq \varepsilon$ for a given positive number ε. Call

$$h_\varepsilon(y) := \sup\{h(y) - \varepsilon, 0\}.$$

It is easy to check that

$$\| h_\varepsilon - g \|_\infty \leqq 2\varepsilon, \quad 0 \leqq h_\varepsilon \leqq g \ \text{ on } Y \qquad \text{and} \quad h_\varepsilon \,\big|\, G \setminus F = 0.$$

Since $L_{F,Q}(h_\varepsilon) \leqq L_{G,Q}(h_\varepsilon)$ for any $Q \in \mathscr{D}$ with $Q_0 \subset Q$, it turns out that $\lambda_F(h_\varepsilon) \leqq \lambda_G(h_\varepsilon)$, and hence $\lambda_F(g) \leqq \lambda_G(g)$, because ε is arbitrary. In a similar manner we find that each λ_G is carried on G. Thus $(\lambda_G : G \in F_p)$ is a consistent system with base point p, whose terminal measure is equal to μ by (4).

Next, we verify a condition for μ to allow the system on $C_R[Q]$ as above. Let B be the closed subalgebra of A generated by a fixed $Q \in \mathscr{D}$ and constants. The letter X denotes the maximal ideal space of B. There exists the continuous map τ of Y into X such that $f(y) = f(\tau(y))$ for all $f \in B$ and $y \in Y$. Call

$$K = \tau(Y) = \{\tau(y) : y \in Y\}.$$

The map τ induces the isometric lattice isomorphism τ_* between $C_R[Q]$ and $C_R(K)$, which satisfies $\tau_* g = g \circ \tau$ for any $g \in C_R(K)$. Note that τ_* preserves the functions of the form:

$$\sup\{c_j \log |f_j| : R \ni c_j \geqq 0, f_j \in B\}.$$

A Jensen measure μ for $p \in Y$ associated with A defines a unique Jensen measure ν to B which represents the evaluation functional at $\tau(p) \in K$ and such that $\mu \circ \tau_* = \nu$. Clearly ν is carried on K.

The following two lemmas give a criterion for μ to satisfy the premise of Lemma 3.1. We denote by $F_{\tau(p)}(K)$ the totality of compact sets in K that contain $\tau(p)$.

LEMMA 3.2. *Suppose K carries a system $(\lambda_G : G \in F_{\tau(p)}(K))$ of Jensen measures for $\tau(p)$ such that*

(1) *each λ_G is supported on G;*

(2) *if $F \subset G$, we have $\lambda_F \prec \lambda_G$;*

(3) *inclusion $F \subset G$ implies $\lambda_F(g) \leqq \lambda_G(g)$ for a nonnegative $g \in C_R(K)$ with $g \mid G \setminus F = 0$;*

(4) $\lambda_K = \nu$.

Then there exists a system $(L_{G,Q} : G \in F_p)$ of positive linear forms on $C_R[Q]$ which possesses four properties in Lemma 3.1.

REMARK. If K is an arbitrary B-convex subset of X, the above system gives a consistent system associated with B_K.

PROOF. Call $L_{G,Q} = \lambda_{\tau(G)} \circ \tau_*^{-1}$ and we require that $(L_{G,Q} : G \in F_p)$ is the desired system. If $g \in C_R[Q]$ vanishes on G, $\tau_*^{-1} g$ is identically zero on $\tau(G)$. Hence by (1)

$$L_{G,Q}(g) = \lambda_{\tau(G)}(\tau_*^{-1} g) = 0.$$

Let $k \in C_R[Q]$ be any function of the form:

$$\sup\{c_j \log |f_j| : R \ni c_j \geqq 0, f_j \in Q\}.$$

Note that $\tau_*^{-1} k$ takes the same form on K. Hence for any $F, G \in F_p$ with $F \subset G$ we get by $\tau(F) \subset \tau(G)$

$$\tau_*^{-1} k(\tau(p)) \leqq \lambda_{\tau(F)}(\tau_*^{-1} k) \leqq \lambda_{\tau(G)}(\tau_*^{-1} k),$$

or equivalently

$$k(p) \leqq L_{F,Q}(k) \leqq L_{G,Q}(k).$$

Furthermore if $g \in C_R[Q]$ is nonnegative and vanishes on $G \setminus F$, it turns out that

$$L_{F,Q}(g) = \lambda_{\tau(F)}(\tau_*^{-1} g) \leqq \lambda_{\tau(G)}(\tau_*^{-1} g) = L_{G,Q}(g),$$

since $\tau_*^{-1} g$ vanishes on $\tau(G) \setminus \tau(F)$.

The final property is immediate from the definition of ν.

LEMMA 3.3. *Suppose that for each closed neighbourhood V of K there exists* *a consistent system* $(\lambda_{G,V}: G \in F_{\tau(p)})$ *of* B *such that* $\lambda_{V,V} = \nu$. *Then there exists a system* $(\lambda_G: G \in F_{\tau(p)}(K))$ *of Jensen measures for* $\tau(p)$ *which satisfies four conditions of Lemma* 3.2.

PROOF. Since B is separable, there is a metric ρ on the maximal ideal space X of B, which is compatible with the initial topology on X. For each positive integer $n \in \mathbf{N}$ and $G \in F_{\tau(p)}(K)$ call

$$G_n = \{z \in X: \operatorname{dist}(z, G) \leqq 1/n\}, \qquad \operatorname{dist}(z, G) = \inf_{w \in G} \{\rho(z, w)\}.$$

Also let I be an ultra filter on $\mathbf{N}$ which includes the filter base $(\{m \in \mathbf{N}: m \geqq n\})_{n \in \mathbf{N}}$.

Putting $\lambda_G = \lim_I \lambda_{G_n, K_n}$, we require that $(\lambda_G: G \in F_{\tau(p)}(K))$ is the desired system. Since each λ_{G_n, K_n} is a Jensen measure for $\tau(p)$ carried on G_n, the measure λ_G is a Jensen measure for $\tau(p)$ supported on G.

Next, suppose a pair $F, G \in F_{\tau(p)}(K)$ satisfies $F \subset G$. It follows from $F_n \subset G_n$ that $\lambda_{F_n, K_n} \prec \lambda_{G_n, K_n}$, and hence $\lambda_F \prec \lambda_G$. Moreover, let $g \in C_R(K)$ be nonnegative and vanish on $G \setminus F$. We may suppose $g \in C_R(X)$. Call $g_\varepsilon = \sup\{g - \varepsilon, 0\}$, where ε is any positive number. It is easily seen that the closed support of g_ε is away from $G \setminus F$ with distance $r > 0$. This implies that g_ε vanishes on $G_n \setminus F_n$ whenever $n > 1/r$. Specifically we get, for such an n

$$\lambda_{F_n, K_n}(g_\varepsilon) \leqq \lambda_{G_n, K_n}(g_\varepsilon), \qquad \text{so that } \lambda_F(g_\varepsilon) \leqq \lambda_G(g_\varepsilon).$$

This yields $\lambda_F(g) \leqq \lambda_G(g)$, since $g_\varepsilon \to g$ uniformly on K as $\varepsilon \to 0$.

Finally the assumption $\lambda_{K_n, K_n} = \nu$ yields $\lambda_K \doteq \nu$.

In order to show that the neighbourhood V above allows the system in question, we realize B as an algebra on a compact set in $\mathbf{C}^n$. Let $Q = (f_1, \ldots, f_n)$ be an arbitrary n-tuple from A. K_Q is the compact subset of $\mathbf{C}^n$ defined by $K_Q = \{(f_1(y), \ldots, f_n(y)): y \in Y\}$. We denote by B_Q the closure in $C(X_Q)$ of analytic polynomials on $\mathbf{C}^n$, where X_Q is the polynomially convex hull of K_Q. B_Q is of course identical with the closed subalgebra of A generated by constants and Q. For any Jensen measure μ for $p \in Y$ associated with A, μ_Q denotes the Jensen measure for $p_Q \in K_Q$, which is induced from μ to the algebra B_Q. Note that μ_Q is carried on K_Q. Also $\pi_{n,m}$ $(n < m)$ is the canonical projection of $\mathbf{C}^m$ onto $\mathbf{C}^n$, i.e.

$$\pi_{n,m}((z_1, \ldots, z_n, z_{n+1}, \ldots, z_m)) = (z_1, \ldots, z_n).$$

Lemma 3.4. *For any n-tuple Q in A, μ_Q is the terminal measure of a consistent system on X_Q.*

Proof. We may suppose $X_Q \subset U^n$. Then μ_Q is a Jensen measure of $A(U^n)$. By Proposition 2.6 there exists a consistent system $(\lambda_G : G \in F_{p_Q})$ whose terminal measure is equal to μ_Q. Since μ_Q is carried on K_Q, each λ_G is supported on X_Q by Proposition 2.7.

Let N be any compact neighbourhood of X_Q in $\overline{U^n}$. By (c) in the definition of consistency we see that

$$\lambda_N = \lambda_N \mid N^0 \leqq \mu_Q, \qquad \text{while } \lambda_N(1) = \mu_Q(1) = 1.$$

This implies $\lambda_N = \mu_Q$. Hence it follows from Lemma 3.3 that there is a system $(\lambda_G : G \in F_{p_Q}(X_Q))$ which satisfies four properties in Lemma 3.2 with μ_Q in place of ν. As pointed out in the remark on Lemma 3.2, this system is a consistent system associated with $B_Q = A(U^n)_{X_Q}$, whose terminal measure is identical with μ_Q.

Theorem 3.5. *Let A be an arbitrary uniform algebra with the maximal ideal space Y. Any Jensen measure μ for an arbitrary point $p \in Y$ is the terminal measure of a consistent system associated with A.*

Proof. Let Q be any n-tuple in A. In view of the preparatory survey, we have only to show that any compact neighbourhood V of K_Q in X_Q allows a consistent system $(\lambda_{G,V} : G \in F_{p_Q})$ of B_Q such that $\lambda_{V,V} = \mu_Q$. By Arens–Calderón's lemma we can pick up a $(m-n)$-tuple $(f_{n+1}, \ldots, f_m)$ from A such that $\pi_{n,m}(X_R) \subset V$ ([2], cf. [10]), where

$$R = (f_1, \ldots, f_n, f_{n+1}, \ldots, f_m) \quad \text{and} \quad Q = (f_1, \ldots, f_n).$$

Also by Lemma 3.4 there exists a consistent system $(\tilde{\lambda}_G : G \in F_{p_R})$, associated with B_R, whose terminal measure is equal to μ_R. For any compact set G with $p_Q \in G \subset X_Q$, call

$$G_R = \pi_{n,m}^{-1}(G) \cap X_R \quad \text{and} \quad \lambda_{G,V}(g) = \tilde{\lambda}_{G_R}(g \circ \pi_{n,m} \mid X_R), \qquad \forall g \in C_R(X_Q).$$

Since $\pi_{n,m} \mid X_R$ is a continuous map of X_R into $V \subset X_Q$, the measure $\lambda_{G,V}$ is well-defined, and is a probability measure on V.

We require that $(\lambda_{G,V} : G \in F_{p_Q})$ is a desired consistent system with respect to B_Q. It follows from $V_R = X_R$ that $\lambda_{V,V} = \mu_Q$, since $(\mu_R)_Q = \mu_Q$ if we regard Q as an n-tuple in B_R.

If $g \in C_R(X_Q)$ is B_Q-subharmonic, then $g \circ \pi_{n,m} \mid X_R$ is B_R-subharmonic. So we obtain that each $\lambda_{G,V}$ is a Jensen measure for $p_Q = (p_R)_Q$, and that $\lambda_{F,V} < \lambda_{G,V}$ if $F \subset G$. Also it is easy to check that $\lambda_{G,V}$ is carried on G.

Finally if $F \subset G$, $F, G \in F_{p_Q}$, and if $g \in C_R(X_Q)$ is a nonnegative function with $g \mid F \setminus G = 0$, the function $g \circ \pi_{n,m} \mid X_R$ vanishes on $G_R \setminus F_R$. This yields

$$\lambda_{F,V}(g) = \tilde{\lambda}_{F_R}(g \circ \pi_{n,m} \mid X_R) \leqq \tilde{\lambda}_{G_R}(g \circ \pi_{n,m} \mid X_R) = \lambda_{G,V}(g).$$

Thus we conclude that $(\lambda_{G,V}: G \in F_{p_Q})$ is the desired consistent system on V.

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